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Refined factorizations of solvable potentials

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Abstract. A generalization of the factorization technique is shown to be a powerful algebraic tool to discover further properties of a class of integrable systems in quantum mechanics. The method is applied in the study of radial oscillator, Morse and Coulomb potentials to obtain a wide set of raising and lowering operators, and to show clearly the connection that links these systems.

1. Introduction

This paper proposes an improvement in the well known factorization method [1]. We shall show that its systematic application on three relevant physical examples uncovers, in a direct form, the involved algebraic relationships and gives, at the same time, a richer structure than the usual factorization method. In this section we recall some basic facts of the standard factorization, as can be found in [2], mainly to fix the notation. Afterwards, in the spirit of [3], we will formalize the more general factorizations and the way they depart from the conventional ones.

Let us consider a sequence of stationary one-dimensional Schrödinger equations, labelled by an integer number ℓ, written in the form

$$H^\ell \psi_n^\ell \equiv \left\{ -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V^\ell(x) \right\} \psi_n^\ell(x) = E_n^\ell \psi_n^\ell(x) \tag{1}$$

where the constants $\hbar$ and m have been conveniently reabsorbed. If such a set (or 'hierarchy') of Hamiltonians can be expressed as

$$H^\ell = X_\ell^+ X_\ell^- - q(\ell) = X_{\ell-1}^- X_{\ell-1}^+ - q(\ell-1) \tag{2}$$

$$X_\ell^\pm = \mp \frac{\mathrm{d}}{\mathrm{d}x} + w_\ell(x) \tag{3}$$

$w_\ell(x)$ being functions and $q(\ell)$ constants, then we will say that it admits a factorization. From (3) we have that $X_\ell^\pm$ are Hermitian conjugate to each other, $(X_\ell^-)^\dagger = X_\ell^+$, with respect to the usual inner product of the Schrödinger equation, which is consistent with the factorization (2) and the Hermiticity of H^ℓ.

We shall focus our interest on studying the discrete spectrum of each Hamiltonian, so we further impose that the equation

$$X_\ell^- \psi_\ell^\ell(x) = 0 \tag{4}$$

will determine the ground state of H^ℓ if it exists. Of course, many other properties related to the continuous spectrum (see [4] and works on this matter by Yale's group [5]) can also be derived with the help of factorizations, but they are out of our present scope. Some consequences that can be derived immediately from the previous conditions are enumerated below.

(i) Spectrum. Let ψ_ℓ^ℓ be the ground state of H^ℓ as stated in (4), then its energy is $E_\ell^\ell = -q(\ell)$. When there are excited bound states ψ_n^ℓ, with $n = \ell, \ell+1, \ldots$, their energy is given by $E_n^\ell = -q(n)$. Therefore, $-q(\ell)$ should be an increasing function on ℓ.

(ii) Eigenfunctions. It is straightforward to check, for each ℓ, the intertwining relations

$$H^\ell X_\ell^+ = X_\ell^+ H^{\ell+1} \qquad H^{\ell+1} X_\ell^- = X_\ell^- H^\ell. \tag{5}$$

Equation (5) displays the well known partnership, usual in the factorization scheme, between the potentials with parameters ℓ and $\ell+1$, in general associated with centrifugal terms (for a more detailed discussion see [6]). Now, let us designate by $\mathcal{H}^\ell = \langle\{\psi_n^\ell\}_{n\geqslant\ell}\rangle$ the Hilbert space spanned by the bound states of H^ℓ, for $\ell \in \mathbb{Z}$. Then, due to (5) the operators $X_\ell^\pm$ link these spaces as

$$\begin{array}{ll} X_\ell^- : \mathcal{H}^\ell \to \mathcal{H}^{\ell+1} & X_\ell^+ : \mathcal{H}^{\ell+1} \to \mathcal{H}^\ell \\ X_\ell^- \psi_n^\ell(x) \propto \psi_n^{\ell+1}(x) & X_\ell^+ \psi_n^{\ell+1}(x) \propto \psi_n^\ell(x). \end{array} \tag{6}$$

Remark that the action of $X_\ell^\pm$ preserves the label n, that is, they connect eigenfunctions with the same energy E_n^ℓ. If the eigenfunctions are normalized we can be more explicit: up to an arbitrary phase factor,

$$\begin{array}{ll} X_\ell^- \psi_n^\ell(x) = \sqrt{q(\ell)-q(n)}\, \psi_n^{\ell+1}(x) & n \geqslant \ell \\ X_\ell^+ \psi_n^{\ell+1}(x) = \sqrt{q(\ell)-q(n)}\, \psi_n^\ell(x) & n > \ell. \end{array} \tag{7}$$

Similar considerations would also apply if the ground states were defined through X^+ and, depending on each particular problem, we then use one of the following notations:

$$X_{\ell-1}^+(r)\psi_{-\ell}^\ell = 0 \qquad \text{if} \quad \ell \leqslant 0 \tag{8}$$

or

$$X_{\ell-1}^+(r)\psi_\ell^\ell = 0 \qquad \text{if} \quad \ell \geqslant 0 \tag{9}$$

where $-q(\ell)$ must be a decreasing function of ℓ. We shall also have the opportunity to illustrate this situation in some examples in the next sections.

Now, it is natural to define a set of free-index linear operators $\{X^\pm, L\}$ acting on the direct sum of the Hilbert spaces $\mathcal{H} \equiv \oplus_\ell \mathcal{H}^\ell$, by means of

$$X^- \psi_n^\ell := X_\ell^- \psi_n^\ell \qquad X^+ \psi_n^\ell := X_{\ell-1}^+ \psi_n^\ell \qquad L\psi_n^\ell := \ell\psi_n^\ell \tag{10}$$

where one must have in mind (6) and (7). That is, the operators $X^\pm$ act on each function $\psi_n^\ell(x)$ by means of the differential operators (3) changing ℓ into $\ell \mp 1$. The action on any other vector of $\mathcal{H}$ can be obtained from (10) by linearization, but we shall never need it. At this moment we are not in a situation to guarantee that the space $\mathcal{H}$ is invariant under this action (it might happen that the action of $X^\pm$ on $\mathcal{H}$ could lead us to the continuous spectrum, or even to an unphysical eigenfunction), but we postpone this problem to the examples of section 3.

Taking into account our definitions (10), it is straightforward to arrive at the following commutators:

$$[L, X^\pm] = \mp X^\pm \qquad [X^+, X^-] = q(L) - q(L-1). \tag{11}$$

It is clear that, in general, the set of operators $\{X^\pm, L\}$ does not close a Lie algebra; relations (11) only allow us to speak formally of an associative algebra. There are many aspects of the conventional factorizations above characterized which can be modified, mainly with the objective of being applicable to a wider class of systems (see for example [7]). However, in this paper we are interested in going deeply into the possibilities of the method itself, investigating a class of systems where the usual factorization can already be applied, with the goal of obtaining additional information. With this aim, we shall stress here two points that will be useful in the next sections.

First, we shall assume that the operators $X_\ell^\pm$ do not have to take necessarily the form given in (3). In particular, if we have a family of invertible operators D_ℓ and define $Y_\ell^+ = X_\ell^+ D_\ell^{-1}$, $Y_\ell^- = D_\ell X_\ell^-$, we will also have

$$H^\ell = X_\ell^+ X_\ell^- - q(\ell) = Y_\ell^+ Y_\ell^- - q(\ell). \tag{12}$$

The new factor D_ℓ may be a function (which would add nothing specially new) but also a local operator, i.e. an operator acting on wavefunctions in the form $D_\ell \psi(x) = \psi(g_\ell(x))$, where $g_\ell(x)$ is a bijective real function. An operator of this type (which has already been used in [8]), is given by the dilation

$$D(\mu)\psi(x) = \psi(\mu x) \qquad \mu > 0. \tag{13}$$

Second, an eigenvalue equation can be characterized by more than one label (this fact has been also explored by Barut *et al* [9] in another context). In the next section we shall deal with two real labels; this will enable us to have more possible ways to factorize the Hamiltonian hierarchy, and the sequence of labels will not be limited (essentially) to the integers, but it will be constituted by a lattice of points in $\mathbb{R}^2$. This increasing of factorizations will reflect itself in a larger algebra of free-index operators. In particular, among them, there can be lowering and raising operators for each Hamiltonian, which can never be obtained by the conventional factorization method. Section 3 will illustrate how our general method works when it is applied to three well known potentials: radial oscillator, Morse, and radial Coulomb. The results for the radial oscillator are well known [3, 10], but we have included them here to show clearly the differences with the application of a naive factorization method. The results concerning Morse and radial Coulomb potentials are new, to our knowledge, and are important in order to establish a link with the algebra underlying the radial oscillator potential. For each of these potentials we shall see that the results obtained can be used to recover, as special cases, those corresponding to the standard factorizations. Some comments and remarks will end the paper.

2. Refined factorizations

Once the spectrum E_n^ℓ of the hierarchy H^ℓ is known, we propose a somewhat more general factorization of the eigenvalue equations than that already displayed in (2), as follows:

$$h_{n,\ell}(x)[H^\ell - E_n^\ell] = B_{n,\ell} A_{n,\ell} - \phi(n,\ell) = A_{\tilde{n},\tilde{\ell}} B_{\tilde{n},\tilde{\ell}} - \phi(\tilde{n},\tilde{\ell}). \tag{14}$$

This must be understood as a series of relationships valid for a class of allowed values of the parameters $(n,\ell) \in \mathbb{R}^2$. Here $B_{n,\ell}$ and $A_{n,\ell}$ are first-order differential operators in the wider sense specified in the previous section, $h_{n,\ell}(x)$ denote functions and $\phi(n,\ell)$ are constants. The $(\tilde{n},\tilde{\ell})$ values depend on (n,ℓ), i.e. $(\tilde{n},\tilde{\ell}) = F(n,\ell)$, $F : \mathbb{R}^2 \to \mathbb{R}^2$ being an invertible map defined on a certain domain. The iterated action of F or F^{-1} on a fixed initial point $(n_0,\ell_0) \in \mathbb{R}^2$ originates a sequence of points in $\mathbb{R}^2$ that will play a role similar to the integer sequence ℓ in $\mathbb{R}$ for the usual factorizations. In principle, the points (n,ℓ) obtained by this new approach can take integer values for both arguments, but we do not discard *a priori* other possibilities.

The problem of finding solutions to this kind of factorization becomes more involved because we have additional functions $h_{n,\ell}(x)$ to be determined. Nevertheless, an important and immediate consequence of (14) is that the operators $B_{n,\ell}$ and $A_{n,\ell}$ share properties similar to (5) for their analogues $\{X_\ell^\pm\}$:

$$\begin{aligned} &[h_{\hat{n},\hat{\ell}}(x)(H^{\hat{\ell}} - E_{\hat{n}}^{\hat{\ell}})]A_{n,\ell} = A_{n,\ell}[h_{n,\ell}(x)(H^\ell - E_n^\ell)] \\ &B_{n,\ell}[h_{\hat{n},\hat{\ell}}(x)(H^{\hat{\ell}} - E_{\hat{n}}^{\hat{\ell}})] = [h_{n,\ell}(x)(H^\ell - E_n^\ell)]B_{n,\ell} \end{aligned} \tag{15}$$

where $F(\hat{n}, \hat{\ell}) = (n, \ell)$. Therefore, using the same notation as in (6),

$$\begin{aligned} &A_{n,\ell} : \mathcal{H}^\ell \to \mathcal{H}^{\hat{\ell}} &&B_{n,\ell} : \mathcal{H}^{\hat{\ell}} \to \mathcal{H}^\ell \\ &A_{n,\ell}\psi_n^\ell(x) \propto \psi_{\hat{n}}^{\hat{\ell}}(x) &&B_{n,\ell}\psi_{\hat{n}}^{\hat{\ell}}(x) \propto \psi_n^\ell(x). \end{aligned} \tag{16}$$

In this case, the most relevant differences with respect to the usual factorizations are the following.

(i) $B_{n,\ell}$, $A_{n,\ell}$ in general do not preserve the energy eigenvalue; they may change both labels n and ℓ.

(ii) $A_{n,\ell}$ does not act on the whole space $\mathcal{H}^\ell$; it acts just on the eigenfunction $\psi_n^\ell(x) \in \mathcal{H}^\ell$ (the same can be said of $B_{n,\ell}$ with respect to $\psi_{\hat{n}}^{\hat{\ell}}(x) \in \mathcal{H}^{\hat{\ell}}$).

When $n = \hat{n}$ and $h_{n,\ell}(x) = 1$, we recover the conventional case with $B_{n,\ell}$, $A_{n,\ell}$ playing the role of X_ℓ^+, X_ℓ^-, respectively. However, the Hermiticity properties for the general case are lost because the product $B_{n,\ell}A_{n,\ell}$ gives not the Hamiltonian operator alone, but the Hamiltonian multiplied by a non-constant factor.

We can define the free-index operators $\{A, B, L, N\}$ as we did in (10), where the latter is defined by $N\psi_n^\ell = n\psi_n^\ell$. They satisfy the following commutation rules:

$$\begin{aligned} &[L, B] = B(\tilde{L} - L) &&[N, B] = B(\tilde{N} - N) &&[B, A] = \phi(N, L) - \phi(\tilde{N}, \tilde{L}) \\ &[L, A] = (L - \tilde{L})A &&[N, A] = (N - \tilde{N})A &&[N, L] = 0 \end{aligned} \tag{17}$$

where $(\tilde{N}, \tilde{L}) = F(N, L)$. As the operators L and N commute, their eigenvalues are used to label the common eigenfunctions $\psi_n^\ell(x)$. We must also notice that the equation $A_{n,\ell}\psi_n^\ell(x) = 0$ (or $B_{n,\ell}\psi_{\hat{n}}^{\hat{\ell}}(x) = 0$) does not necessarily give an eigenfunction of H^ℓ (or $H^{\hat{\ell}}$); this happens to be the case only when $\phi(n, \ell) = 0$. In principle, there can be many solutions to (14), but we will pay attention to those which are more 'elementary', in the sense that they cannot be obtained by the composition of other already known solutions.

3. Applications

3.1. Radial oscillator potential

As usual, the Hamiltonian of the two-dimensional harmonic oscillator includes the effective radial potential $V^\ell(r) = r^2 + \frac{(2\ell+1)(2\ell-1)}{4r^2}$, where $\ell = 0, 1 \ldots$ represents the angular momentum. The related stationary Schrödinger equation has discrete eigenvalues denoted according to the following convention:

$$E_n^\ell = 2n + 2 \qquad n = 2\nu + \ell \qquad \nu = 0, 1, \ldots.$$

It can be factorized in two ways according to our general scheme:

$$\frac{-1}{4}[H^\ell - E_n^\ell] = \frac{1}{4}\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - r^2 - \frac{(2\ell+1)(2\ell-1)}{4r^2} + E_n^\ell\right] = B_{n,\ell}^i A_{n,\ell}^i - \phi^i(n, \ell) \qquad i = 1, 2 \tag{18}$$

with $\phi^i(n,\ell)$ given by

$$\phi^1(n,\ell) = -\tfrac{1}{2}(n+\ell+2) \qquad \phi^2(n,\ell) = -\tfrac{1}{2}(n-\ell+2) \tag{19}$$

and where the action on the parameters associated with each factorization is given respectively by the functions

$$(n,\ell) = F^1(n+1,\ell+1) \qquad (n,\ell) = F^2(n+1,\ell-1). \tag{20}$$

This can also be written in an easier notation,

$$\begin{array}{ll} A^1 : (n,\ell) \to (n+1,\ell+1) & A^2 : (n,\ell) \to (n+1,\ell-1) \\ B^1 : (n+1,\ell+1) \to (n,\ell) & B^2 : (n+1,\ell-1) \to (n,\ell). \end{array} \tag{21}$$

The explicit form of these intertwining operators is

$$\begin{cases} A^1_{n,\ell}(r) = \dfrac{1}{2}\left[\dfrac{\mathrm{d}}{\mathrm{d}r} - r - \left(\ell+\dfrac{1}{2}\right)\dfrac{1}{r}\right] \\ B^1_{n,\ell}(r) = \dfrac{1}{2}\left[\dfrac{\mathrm{d}}{\mathrm{d}r} + r + \left(\ell+\dfrac{1}{2}\right)\dfrac{1}{r}\right] \end{cases} \qquad \begin{cases} A^2_{n,\ell}(r) = \dfrac{1}{2}\left[\dfrac{\mathrm{d}}{\mathrm{d}r} - r + \left(\ell-\dfrac{1}{2}\right)\dfrac{1}{r}\right] \\ B^2_{n,\ell}(r) = \dfrac{1}{2}\left[\dfrac{\mathrm{d}}{\mathrm{d}r} + r - \left(\ell-\dfrac{1}{2}\right)\dfrac{1}{r}\right]. \end{cases} \tag{22}$$

Observe that in this case, as $h_{n,\ell}(r)$ is a constant, we are able to implement also the Hermiticity properties $(A^i)^\dagger = -B^i$. The non-vanishing commutation rules for the free-index operators $\{N, L, A^i, B^i; i = 1, 2\}$ are shown to be, in agreement with (17),

$$\begin{array}{lll} [L, B^i] = (-1)^i B^i & [N, B^i] = -B^i & [A^i, B^i] = 1 \\ [L, A^i] = -(-1)^i A^i & [N, A^i] = A^i. & \end{array} \tag{23}$$

These commutators correspond to two independent boson algebras with N and L being a linear combination of their number operators. Formally we can extend the values of ℓ to include the negative integers. This is physically appealing because in two space dimensions (only!) ℓ represents the L_z-component of angular momentum, so it could take negative integer values. Of course the extension $\psi_n^{-\ell}(r) := \psi_n^{\ell}(r)$ proposed above is consistent with such an interpretation: (i) the radial components for opposite L_z-values have to coincide, and (ii) the potential V^ℓ is invariant under the interchange $\ell \to -\ell$. With this convention, the Hilbert space $\mathcal{H}$ of bound states is invariant under the action of the operators $\{N, L, A^i, B^i; i = 1, 2\}$, so it constitutes the support for a lowest-weight irreducible representation for the algebra (23) based on the fundamental state $\psi_{n=0}^{\ell=0}$.

It is worth noticing that, taking into account (21), the compositions A^1A^2 and B^1B^2 constitute, respectively, the lowering and raising operators for each Hamiltonian H^ℓ, while the pair $\{A^1B^2, A^2B^1\}$ connects states of different Hamiltonians H^ℓ with the same energy, changing only the label ℓ.

We shall compare briefly the above results with the conventional factorizations of the two-dimensional radial oscillator potential [3, 10]. It is well known that there are two such factorizations, which we will write in the form

$$(a) \quad X_\ell^+ X_\ell^- - q_x(\ell) = H_x^\ell = H^\ell - 2\ell \qquad (b) \quad Z_\ell^+ Z_\ell^- - q_z(\ell) = H_z^\ell = H^\ell + 2\ell \tag{24}$$

with $H^\ell = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + r^2 + \frac{(2\ell+1)(2\ell-1)}{4r^2}$. Then we have the following identification.

Case (*a*).

(1) *Operators.* $X_\ell^+ = -2B^1_{n,\ell}$, $X_\ell^- = 2A^1_{n,\ell}$, $q_x(\ell) = 4\ell - 2$.
(2) *Ground states.* $X^+_{\ell-1}\psi^\ell_{-\ell} = 0$, $\ell \leqslant 0$.
(3) *Energy eigenvalues.* $E_n^\ell = 4n + 2$, with $n \in \mathbb{Z}^+$ and $n \geqslant -\ell$.

We have used a notation in agreement with (8).

Case (*b*).

(1) *Operators.* $Z^+_\ell = -2A^2_{n-1,\ell+1}$, $Z^-_\ell = 2B^2_{n-1,\ell+1}$, $q_z(\ell) = -4\ell - 2$.
(2) *Ground states.* $Z^-_\ell \psi^\ell_\ell = 0$, $\ell \geqslant 0$.
(3) *Energy eigenvalues.* $E^\ell_n = 4n + 2$, with $n \in \mathbb{Z}^+$ and $n \geqslant \ell$.

Therefore, as there is a correspondence between the results of the conventional factorization and ours, one might conclude the total equivalence of both treatments. However, the conventional factorizations make use of two Hamiltonian hierarchies, H^ℓ_x and H^ℓ_z, whose terms differ in a constant 4ℓ, while the new factorizations use only one H^ℓ. If we want both factorizations (a) and (b) to be valid inside the same hierarchy, it is necessary to adopt the properties of our approach in the following sense: either the operators $X^\pm_\ell$ or $Z^\pm_\ell$ (or both pairs) must change not only the quantum number ℓ but also n. By means of this example we have shown that the factorizations presented here are quite useful, providing directly a more natural viewpoint than the usual ones.

3.2. Morse potential

We have eigenvalue Schrödinger equations for the whole real line with the potentials

$$V^\ell(x) = \left(\frac{\alpha}{2}\right)^2 (\mathrm{e}^{2\alpha x} - 2(\ell+1)\mathrm{e}^{\alpha x}) \qquad \alpha > 0 \qquad \ell \geqslant 0. \tag{25}$$

Often in the literature [11] the Morse potentials are written $V(y) = A(\mathrm{e}^{-2\alpha y} - 2\mathrm{e}^{-\alpha y})$. This form can be reached from (25) by the simple change of variable $x = -y + k$, with $e^{\alpha k} = \ell + 1$. The energy eigenvalues can be expressed as

$$E^\ell_n = -\frac{\alpha^2}{4} n^2 \qquad n = \ell - 2\nu > 0 \qquad \nu = 0, 1, 2, \ldots. \tag{26}$$

In order to have bound states it is necessary to impose the restriction $\ell > 0$; the critical value $\ell = 0$ has an special limiting character, and it is convenient to take it into account as we shall see later. According to (26), the eigenfunctions ψ^ℓ_n are characterized by labels satisfying $n \leqslant \ell$; this means that the ground states will be defined through (9). There are two new factorizations

$$\frac{-\mathrm{e}^{-\alpha x}}{\alpha^2}[H^\ell - E^\ell_n] = B^i_{n,\ell}(x) A^i_{n,\ell}(x) - \phi^i(n,\ell) \qquad i = 1, 2 \tag{27}$$

with $\phi^i(n,\ell)$ given by

$$\phi^1(n,\ell) = -\tfrac{1}{2}(\ell + n + 2) \qquad \phi^2(n,\ell) = -\tfrac{1}{2}(\ell - n + 2) \tag{28}$$

and the action on the parameters (n,ℓ) for each factorization by the functions

$$(n,\ell) = F^1(n+1, \ell+1) \qquad (n,\ell) = F^2(n-1, \ell+1). \tag{29}$$

The explicit form of the intertwining operators (27) is

$$\begin{cases} B^1_{n,\ell}(x) = \dfrac{\mathrm{e}^{-\alpha x/2}}{\alpha}\dfrac{\mathrm{d}}{\mathrm{d}x} + \dfrac{1}{2}\mathrm{e}^{\alpha x/2} + \dfrac{n+1}{2}\mathrm{e}^{-\alpha x/2} \\ A^1_{n,\ell}(x) = \dfrac{\mathrm{e}^{-\alpha x/2}}{\alpha}\dfrac{\mathrm{d}}{\mathrm{d}x} - \dfrac{1}{2}\mathrm{e}^{\alpha x/2} - \dfrac{n}{2}\mathrm{e}^{-\alpha x/2} \end{cases} \tag{30}$$

$$\begin{cases} B^2_{n,\ell}(x) = \dfrac{\mathrm{e}^{-\alpha x/2}}{\alpha}\dfrac{\mathrm{d}}{\mathrm{d}x} + \dfrac{1}{2}\mathrm{e}^{\alpha x/2} - \dfrac{n-1}{2}\mathrm{e}^{-\alpha x/2} \\ A^2_{n,\ell}(x) = \dfrac{\mathrm{e}^{-\alpha x/2}}{\alpha}\dfrac{\mathrm{d}}{\mathrm{d}x} - \dfrac{1}{2}\mathrm{e}^{\alpha x/2} + \dfrac{n}{2}\mathrm{e}^{-\alpha x/2}. \end{cases} \tag{31}$$

As in the oscillator case we have two pairs of operators that change simultaneously two types of label: one, ℓ, is related to the intensity of the potential, although here it cannot be interpreted as a centrifugal term; the other, n, is directly related to the energy through formula (26). The (non-vanishing) commutators of the free-index operators are ($i = 1, 2$)

$$[L, B^i] = -B^i \qquad [N, B^i] = (-1)^i B^i \qquad [A^i, B^i] = 1$$
$$[L, A^i] = A^i \qquad [N, A^i] = -(-1)^i A^i.$$

Observe that in this case the function $h_{n,\ell}(x) = -\mathrm{e}^{-\alpha x}/\alpha^2$ is not a constant, so the Hermiticity relations among the operators $\{A^i, B^i; i = 1, 2\}$ are spoiled. Let us take $\ell \in \mathbb{Z}^+$, and formally allow for negative n-values in (26), i.e. $\pm n = \ell - 2\nu$; this is admissible because in the operators of (30), (31) we have a symmetry under the change $n \to -n$. Then the Hilbert space $\mathcal{H}$ of bound states enlarged with the (not square integrable) states $\psi^\ell_{n=0}$, $\ell = 0, 1, \ldots,$ will be invariant under the action of all the operators defined in this section. The lowest-weight state role is played by a non-square-integrable wavefunction, $\psi^{\ell=0}_{n=0}$.

We can, of course, build other operators out of the previous ones, changing exclusively one of the labels: the pair $\{A^1A^2, B^1B^2\}$ change ℓ (in +2 or −2 units, respectively), while $\{A^1B^2, A^2B^1\}$ change n (also in +2 or −2 units, respectively). It is interesting to show explicitly the form taken by the former couple:

$$\begin{cases} (B^1B^2)_{n,\ell} = \dfrac{1}{\alpha}\dfrac{\mathrm{d}}{\mathrm{d}x} + \dfrac{1}{2}(\mathrm{e}^{\alpha x} - (\ell + 2)) \\ (A^1A^2)_{n,\ell} = -\dfrac{1}{\alpha}\dfrac{\mathrm{d}}{\mathrm{d}x} + \dfrac{1}{2}(\mathrm{e}^{\alpha x} - (\ell + 2)) \end{cases} \tag{32}$$

where $(A^1A^2)_{n,\ell} = A^1_{n-1,\ell+1}A^2_{n,\ell}$ and $(B^1B^2)_{n,\ell} = B^1_{n,\ell}B^2_{n+1,\ell+1}$, according to the rules of the action of free index operators (16), (10). They can be identified with the usual factorization operators for the Morse Hamiltonians H^ℓ described in the first section in the following way.

(1) *Factorization.* $X^+_{\ell'}X^-_{\ell'} - q(\ell') = H^{2\ell'}$, $\ell' \in \mathbb{Z}^+$.
(2) *Operators.* $X^+_{\ell'} = -\alpha(B^1B^2)_{n,2\ell'}$, $X^-_{\ell'} = -\alpha(A^1A^2)_{n,2\ell'}$, $q(\ell') = \alpha^2(\ell' + 1)^2$.
(3) *Ground states.* $X^+_{\ell'-1}\psi^{\ell'}_{\ell'} = 0$, $\ell' > 0$.
(4) *Energy eigenvalues.* $E^{2\ell'}_{2n'} = -\alpha^2(n')^2$, with $n = 2n'$, $n' \in \mathbb{Z}^+$, and $0 \leqslant n' \leqslant \ell'$.

This time the notation, as mentioned above, is in agreement with (9).

3.3. Radial Coulomb potential

After the separation of the angular variables, the stationary radial Schrödinger equation for the Coulomb potential in two dimensions takes the form

$$H^\ell\psi^\ell_n(r) = \left\{-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(2\ell + 1)(2\ell - 1)}{4r^2} - \frac{2}{r}\right\}\psi^\ell_n(r) = E^\ell_n\psi^\ell_n(r) \tag{33}$$

where the values of the orbital angular momentum are positive integers $\ell = 0, 1, 2, \ldots.$

The computation of the discrete spectrum associated with the bound states of H^ℓ can be easily obtained by means of the conventional factorizations (2) with

$$X^\pm_\ell = \mp\frac{\mathrm{d}}{\mathrm{d}r} - \frac{2\ell + 1}{2r} + \frac{2}{2\ell + 1} \qquad q(\ell) = \frac{-1}{(\ell + 1/2)^2}. \tag{34}$$

Therefore, according to the results quoted in section 1, we have

$$E^\ell_n = -(n + 1/2)^{-2} \qquad n = \ell + \nu \qquad \nu = 0, 1, \ldots. \tag{35}$$

When our method is applied to the hydrogen Hamiltonians H^ℓ of equation (33) with the eigenvalues E^ℓ_n (35), we obtain two independent solutions that read as follows:

$$B^1_{n,\ell}A^1_{n,\ell} + \ell + n + 1 = -\frac{(2n + 1)r}{4}[H^\ell - E^\ell_n] \tag{36}$$

$$B^2_{n,\ell}A^2_{n,\ell} - \ell + n + 1 = -\frac{(2n + 1)r}{4}[H^\ell - E^\ell_n]. \tag{37}$$

The explicit form of the operators $\{A^i, B^i\}_{i=1,2}$, is displayed below:

$$\begin{cases} B^1_{n,\ell}(r) = (2n+1)^{1/2}\left(\dfrac{r^{1/2}}{2}\dfrac{\mathrm{d}}{\mathrm{d}r}+\dfrac{r^{1/2}}{2n+1}+\dfrac{\ell}{2r^{1/2}}\right)c_n^{-1/2}D(c_n) \\ A^1_{n,\ell}(r) = D(c_n{}^{-1})c_n^{1/2}(2n+1)^{1/2}\left(\dfrac{r^{1/2}}{2}\dfrac{\mathrm{d}}{\mathrm{d}r}-\dfrac{r^{1/2}}{2n+1}-\dfrac{2\ell+1}{4r^{1/2}}\right) \end{cases} \tag{38}$$

$$\begin{cases} B^2_{n,\ell}(r) = (2n+1)^{1/2}\left(\dfrac{r^{1/2}}{2}\dfrac{\mathrm{d}}{\mathrm{d}r}+\dfrac{r^{1/2}}{2n+1}-\dfrac{\ell}{2r^{1/2}}\right)c_n^{-1/2}D(c_n) \\ A^2_{n,\ell}(r) = D(c_n{}^{-1})c_n^{1/2}(2n+1)^{1/2}\left(\dfrac{r^{1/2}}{2}\dfrac{\mathrm{d}}{\mathrm{d}r}-\dfrac{r^{1/2}}{2n+1}+\dfrac{2\ell+1}{4r^{1/2}}\right). \end{cases} \tag{39}$$

The symbol $D(\mu)$ in (38), (39) is for the dilation operator (13), and $c_n = \frac{2n+2}{2n+1}$. Thus, in this example we are dealing with general first-order differential operators as explained in section 1. For the first couple $\{A^1, B^1\}$ we have $(\hat{n}, \hat{\ell}) = (n+1/2, \ell+1/2)$, while for the second pair $\{A^2, B^2\}$, $(\hat{n}, \hat{\ell}) = (n+1/2, \ell-1/2)$.

The non-vanishing commutators of the free-index operators are $(i = 1, 2)$

$$[N, B^i] = \frac{-1}{2}B^i \qquad [L, B^i] = (-1)^i\tfrac{1}{2}B^i \qquad [A^i, B^i] = I$$
$$[N, A^i] = \tfrac{1}{2}A^i \qquad [L, A^i] = -(-1)^i\tfrac{1}{2}A^i.$$

In other words, as in the previous examples, we have a set of two independent boson operator algebras. The problem with these operators is that they change the quantum numbers (n, ℓ) in half-units, so that they do not keep inside the sector of physical wavefunctions. To avoid this problem we can build quadratic operators [3, 12] $\{A^iA^j, B^iA^j, B^iB^j\}_{i,j=1,2}$ satisfying this requirement; such second-order operators close the Lie algebra $sp(4,\mathbb{R})$ [5], which includes the subalgebra $su(2)$ (whose generators connect eigenstates with the same energy but different ℓs). It is worth writing down these quadratic operators, that can be written as first-order differential operators making use of the eigenvalue equation (33):

$$\begin{cases} (B^2A^1)_{n,\ell} = \dfrac{(2\ell+1)(2n+1)}{2}\left(\dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}r}+\dfrac{2\ell+1}{4r}-\dfrac{1}{2\ell+1}\right) \\ (B^1A^2)_{n,\ell} = \dfrac{(2\ell+1)(2n+1)}{2}\left(-\dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}r}+\dfrac{2\ell+1}{4r}-\dfrac{1}{2\ell+1}\right). \end{cases} \tag{40}$$

They constitute, up to global constants, the usual factorization operators given in (34): $X^+_\ell \propto (A^1B^2)_{n,\ell}$, $X^-_\ell \propto (A^2B^1)_{n,\ell}$. Another subalgebra is $su(1,1)$ (relating states with the same ℓ but different energies or n). Once we have included the negative ℓ values, as we did for the radial oscillator potential, the space $\mathcal{H}$ is the support for what is called a 'singleton representation' [13] of $so(3,2) \approx sp(4,\mathbb{R})$. There is one lowest-weight eigenvector $\psi^{\ell=0}_{n=0} \in \mathcal{H}$, from which all the representation space is generated by applying raising operators.

4. Conclusions and remarks

We have shown that a refinement of the factorization method along the lines of [3] allows us to study the maximum of relations among the Hamiltonian hierarchies that the conventional factorizations are not able to appreciate. The involved operators obey commutation rules which show the connection existing among the three examples dealt with in this paper: they have the same underlying Lie algebra associated with confluent hypergeometric functions [14]. On other occasions the conventional factorizations have been used in this respect, but we have seen that such an approach is partial and not complete at all. This explains why, in some cases, there is no trace of our 'elementary' factorizations, but, basically, only of the quadratic operators mentioned in the previous sections.

Usually the Hamiltonian hierarchies are obtained from higher-dimensional systems after separation of variables (or by any other way of reduction). Such systems have symmetries that are responsible for their analytical treatment. These symmetries are reflected in the many factorizations that the hierarchies can give rise to by means of the technique we have developed. We have limited our study to $N = 2$ space dimensions for the radial oscillator and Coulomb potentials because they are the simplest cases to deal with. For higher dimensions there appear certain subtleties, in the sense that the Hilbert space $\mathcal{H}$ of bound states is no longer invariant under the involved operators [14], and there may exist accidental degeneracy [15].

Finally, let us mention that we have limited ourselves to some examples (all of them inside the class of shape invariant potentials [16]), but it is clear that the whole treatment is applicable to the remaining Hamiltonians in the classification of Infeld and Hull [1].

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